SPATIAL PROBLEM OF STATIONARY CREEP OF A STOCHASTICALLY INHOMOGENEOUS MEDIUM

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A solution is presented for the spatial problem of stationary nonlinear creep of a medium when it is allowable to neglect the elastic strains. The medium is considered stochastically inhomogeneous so that the stress and strain tensors are random functions of the space coordinates. An analogous problem for a thin stochastically inhomogeneous plate was considered under conditions of the plane state of stress in [1, 2].

It is established that random variations in the mechanical properties of a material are capable of exerting substantial influence on an estimate of the operability of a structure under creep conditions, and the linearization method used in this paper is justifiedly applicable to a sufficiently broad class of real materials. It is also shown that even when the deterministic part of the stress tensor corresponds to the plane stress state, the stress fluctuations in the direction of all three principal axes are quantities of the same order of magnitude.

Let the stress tensor components σ_{ij} satisfy the equilibrium equations

$$\sigma_{ij,j} = 0 \ (i, j = 1, 2, 3), \tag{1}$$

and the strain rate tensor components $\dot{\varepsilon}_{ij}$ the conditions

$$\Lambda_{ijk}\Lambda_{lmn}\dot{\varepsilon}_{km,jn} = 0, \tag{2}$$

which are obtained from the compatibility equations for the strain by differentiation with respect to the time (Λ_{ijk} is the unit asymmetric pseudotensor).

Equations (1) and (2) are closed by the governing relationship which is taken in conformity with nonlinear flow theory:

$$\dot{\varepsilon}_{ij} = A(\sigma_{ij} - (1/3)\delta_{ij}\sigma_{mm}). \tag{3}$$

Here A is a random function describing the stochastic properties of the material

$$\begin{gathered} A = cs^n[1 + \alpha U(x_1, x_2, x_3)], \\ \langle U\rangle = 0,\ \langle U^2\rangle = 1,\ 0 < \alpha < 1,\ s^2 = (1/2)(3\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}). \end{gathered} \tag{4}$$

Fluctuations in the mechanical properties are described by using the random function $U(x_1, x_2, x_3)$ while the number α plays the part of the variation factor for these properties. For real materials α can vary between 0.05 and 0.5. For instance, the variation factor α for ÉI 395 steel, computed from the results of tests borrowed from [3], turns out to be 0.18, while for ÉI 454 steel it is $\alpha = 0.39$.

The problem (1)-(4) is physically and statistically nonlinear, and its approximate solution is constructed in this paper on the basis of a linearization method.

Let the stress tensor be represented in the form of the sum of a deterministic component σ^0_{ij} and the fluctuations σ^*_{ij}:

$$\sigma_{ij} = \sigma^0_{ij} + \sigma^*_{ij},\quad \langle\sigma_{ij}\rangle = \sigma^0_{ij},\quad \langle\sigma^*_{ij}\rangle = 0.$$

The tensor σ^0_{ij} is considered known and can be found as the solution of the appropriate deterministic problem

$$\sigma^0_{ij,j} = 0,\quad \Lambda_{ijk}\Lambda_{lmr}\dot{\varepsilon}^0_{km,jr} = 0,\quad \dot{\varepsilon}^0_{km,jr} = cs^{0n}\left(\sigma^0_{km} - \frac{1}{3}\delta_{km}\sigma^0_{ii}\right)_{,jr}. \tag{5}$$

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The fluctuations σ^*_{ij} are high-frequency perturbations superimposed on the deterministic solution σ^0_{ij}, where the gradients $\sigma^0_{ij,k}$ are considerably less than $\sigma^*_{ij,k}$ in absolute value. Consequently, it can be considered that $\sigma^0_{ij} \approx$ const when studying the fluctuations in a sufficiently small neighborhood of the selected point. For brevity, it is also assumed later that the tensor σ^0_{ij} is reduced to the principal axes. It follows from (1.5) that

$$\sigma^*_{ij,j} = 0. \tag{6}$$

Linearization of the relationships (4) relative to the fluctuations σ^*_{ij} is executed taking into account the possibility of neglecting products of the form $\sigma^*_{ij}\sigma^*_{hl}$,

$$s^n = (s^{02} + s^*)^{n/2} \approx s^{0n} + \frac{n}{2} s^{0n-2} s^*, \tag{7}$$

where

$$s^{02} = \sigma^{02}_{11} + \sigma^{02}_{22} + \sigma^{02}_{33} - \sigma^0_{11}\sigma^0_{22} - \sigma^0_{11}\sigma^0_{33} - \sigma^0_{22}\sigma^0_{33},$$
$$s^* = \sigma^*_{11}p_1 + \sigma^*_{22}p_2 + \sigma^*_{33}p_3, \quad p_\beta = 3\sigma^0_{\beta\beta} - \sigma^0_{ii}$$

(summation is not performed over the Greek subscripts).

The creep strain rates have the following form according to (3) and (7) (products of the form $\alpha U \sigma^*_{ij}$ are discarded)

$$\dot{\varepsilon}_{\beta\beta} = \frac{1}{3} c s^{0n-2} \left[s^{02} p_\beta + s^{02} \left(3\sigma^*_{\beta\beta} - \sigma^*_{ii}\right) + \frac{n}{2} s^* p_\beta + s^{02} \alpha U p_\beta \right],$$
$$\dot{\varepsilon}_{\beta\gamma} = c s^{0n} \sigma^*_{\beta\gamma} \qquad (\beta \neq \gamma). \tag{8}$$

The strain rate tensor fluctuations

$$\dot{\varepsilon}^*_{ij} = \dot{\varepsilon}_{ij} - \dot{\varepsilon}^0_{ij}$$

can be calculated by using (5) and (8)

$$\dot{\varepsilon}^*_{\beta\beta} = \frac{1}{3} c s^{0n-2} \left[s^{02} \left(3\sigma^*_{\beta\beta} - \sigma^*_{ii}\right) + \frac{n}{2} s^* p_\beta + s^{02} \alpha U p_\beta \right],$$
$$\dot{\varepsilon}^*_{\beta\gamma} = c s^{0n} \sigma^*_{\beta\gamma} \qquad (\beta \neq \gamma). \tag{9}$$

If (9) is substituted into the compatibility equations for the strain rate fluctuations $\Lambda_{ijk}\Lambda_{lmn}\dot{\varepsilon}^*_{km,jn} = 0$, then the following relationships can be obtained:

$$\sigma^*_{33,12}(2 + k_3p_3) + \sigma^*_{11,12}(-1 + k_3p_1) + \sigma^*_{22,12}(-1 + k_3p_2) + \alpha U_{,12}p_3 = 3\left(\sigma^*_{13,23} + \sigma^*_{23,13} - \sigma^*_{12,33}\right),$$
$$\sigma^*_{11,13}(-1 + k_2p_1) + \sigma^*_{22,13}(2 + k_2p_2) + \sigma^*_{33,13}(-1 + k_2p_3) + \alpha U_{,13}p_2 = 3\left(\sigma^*_{23,12} + \sigma^*_{12,23} - \sigma^*_{13,22}\right)$$
$$\sigma^*_{11,23}(2 + k_1p_1) + \sigma^*_{22,23}(-1 + k_1p_2) + \sigma^*_{33,23}(-1 + k_1p_3) + \alpha U_{,23}p_1 = 3\left(\sigma^*_{12,13} + \sigma^*_{13,12} - \sigma^*_{23,11}\right),$$
$$\sigma^*_{11,22}(2 + k_1p_1) + \sigma^*_{22,22}(-1 + k_1p_2) + \sigma^*_{33,22}(-1 + k_1p_3) + \sigma^*_{11,11}(-1 + k_2p_1)$$
$$+ \sigma^*_{22,11}(2 + k_2p_2) + \sigma^*_{33,11}(-1 + k_2p_3) + \alpha(U_{,22}p_1 + U_{,11}p_2) = 6\sigma^*_{12,12}, \tag{10}$$
$$\sigma^*_{11,33}(2 + k_1p_1) + \sigma^*_{22,33}(-1 + k_1p_2) + \sigma^*_{33,33}(-1 + k_1p_3) + \sigma^*_{11,11}(-1 + k_3p_1) + \sigma^*_{22,11}(-1 + k_3p_2) + \sigma^*_{33,11}(2 + k_3p_3) + \alpha(U_{,33}p_1$$
$$+ U_{,11}p_3) = 6\sigma^*_{13,13}, \quad \sigma^*_{11,33}(-1 + k_2p_1) + \sigma^*_{22,33}(2 + k_2p_2) + \sigma^*_{33,33}(-1 + k_2p_3) + \sigma^*_{11,22}(-1 + k_3p_1) + \sigma^*_{22,22}(-1 + k_3p_2)$$
$$+ \sigma^*_{33,22}(2 + k_3p_3) + \alpha(U_{,33}p_2 + U_{,22}p_3) = 6\sigma^*_{23,23} \qquad \left(k_i = \frac{np_i}{2s^{02}}\right)$$

(in deriving (10) it is again assumed that σ^0_{ij} are slowly varying functions as compared with σ^*_{ij}).

The linearized problem (6), (10) is later solved in place of (1)-(4).

Let the function $U(x_1, x_2, x_3)$, which is used to give the random perturbation field of the mechanical properties of the material, be homogeneous and isotropic. Then it is representable in the form of a Fourier—Stieltjes integral [4]:

$$U(x_1, x_2, x_3) = \iiint_{-\infty}^{\infty} e^{i\omega_k x_k} d\varphi(\omega_1, \omega_2, \omega_3), \tag{11}$$

where the condition of stochastic orthogonality

$$\langle \overline{d\varphi(\omega_1, \omega_2, \omega_3)\, d\varphi(\omega_1', \omega_2', \omega_3')} \rangle = S_U(\omega_1, \omega_2, \omega_3)\, \delta(\omega_1 - \omega_1')\, \delta(\omega_2 - \omega_2')\, \delta(\omega_3 - \omega_3')\, d\omega_1 d\omega_2 d\omega_3 d\omega_1' d\omega_2' d\omega_3',$$

is satisfied for the random differential $d\varphi(\omega_1, \omega_2, \omega_3)$, where $S_U(\omega_1, \omega_2, \omega_3)$ is the spectral density of the field $U(x_1, x_2, x_3)$, and $\delta(x)$ is the Dirac delta function.

Far from the body boundary, the solution of the problem (6), (10) will also be homogeneous,and it can be sought in the form

$$\sigma^*_{mn} = \int\int\int_{-\infty}^{\infty} e^{i\omega_k x_k} \beta_{mn}(\omega_1, \omega_2, \omega_3)\, d\varphi(\omega_1, \omega_2, \omega_3), \tag{12}$$

where $\beta_{mn}(\omega_1, \omega_2, \omega_3)$ are unknown weight functions.

The unknown functions β_{mn} are calculated from the system of linear equations that is obtained by substituting (11) and (12) into (6) and (10):

$$\omega_1\beta_{11} + \omega_2\beta_{12} + \omega_3\beta_{13} = 0,\ \omega_1\beta_{12} + \omega_2\beta_{22} + \omega_3\beta_{23} = 0,$$
$$\omega_1\beta_{13} + \omega_2\beta_{23} + \omega_3\beta_{33} = 0,$$
$$\omega_1\omega_2[(-1 + k_3p_1)\beta_{11} + (-1 + k_3p_2)\beta_{22} + (2 + k_3p_3)\beta_{33}] - 3\omega_3(\omega_2\beta_{13} + \omega_1\beta_{23} - \omega_3\beta_{12}) = -\omega_1\omega_2\alpha p_3,$$
$$\omega_1\omega_3[(-1 + k_2p_1)\beta_{11} + (2 + k_2p_2)\beta_{22} + (-1 + k_2p_3)\beta_{33}] - 3\omega_2(\omega_1\beta_{23} + \omega_3\beta_{12} - \omega_2\beta_{13}) = -\omega_1\omega_3\alpha p_2,$$
$$\omega_2\omega_3[(2 + k_1p_1)\beta_{11} + (-1 + k_1p_2)\beta_{22} + (-1 + k_1p_3)\beta_{33}] - 3\omega_1(\omega_3\beta_{12} + \omega_2\beta_{13} - \omega_1\beta_{23}) = -\omega_2\omega_3\alpha p_1.$$

In the general case the weight functions have awkward form, consequently, they are not written down.

Now, the various tensor of the random stress field can be determined by using (12):

$$D_{ijkl} = \langle \overline{\sigma^*_{ij}(x_1, x_2, x_3)\, \sigma^*_{kl}(x_1, x_2, x_3)} \rangle = \int\int\int_{-\infty}^{\infty} S_U(\omega_1, \omega_2, \omega_3)\, \beta_{ij}(\omega_1, \omega_2, \omega_3)\, \beta_{kl}(\omega_1, \omega_2, \omega_3)\, d\omega_1 d\omega_2 d\omega_3. \tag{13}$$

After going over the the spherical coordinates

$$\omega_1 = \omega \sin\varphi \cos\theta,\ \omega_2 = \omega \sin\varphi \sin\theta,\ \omega_3 = \omega \cos\varphi$$

and taking into account that [3]

$$D_U = 4\pi \int_0^{\infty} S(\omega)\, \omega^2 d\omega = 1,$$

the integral (13) reduces to the form

$$D_{ijkl} = \int_0^{\pi} R_{ijkl}(\cos\varphi)\, d(\cos\varphi), \tag{14}$$

where $R_{ijkl}(\cos\varphi)$ is a known rational function of $\cos\varphi$. Integrating (14), the variance tensor components can be found explicitly.

As an illustration, the case is considered when $\sigma^0_{11} = \sigma^0_{22} \neq \sigma^0_{33}$. Here the weight functions β_{ij} have the form

$$\beta_{i3} = -\alpha s^0 \frac{(\omega_i\omega_3 - \delta_{i3}\omega^2)(\omega_1^2 + \omega_2^2 - \omega_3^2)}{\Delta} \quad (i = 1, 2, 3),$$
$$\beta_{ij} = -\alpha s^0 \frac{(\omega_i\omega_j - \delta_{ij}\omega^2)\, 2\omega_3^2 + \delta_{ij}\omega_3^2(\omega_1^2 + \omega_2^2 - \omega_3^2)}{\Delta} \quad (i, j = 1, 2), \tag{15}$$

where $\omega^2 = \omega_i\omega_i$, $\Delta = [(1 + n)(\omega_1^2 + \omega_2^2 - \omega_3^2)^2 + (4 + n)\omega_3^2(\omega_1^2 + \omega_2^2)]$.

The formulas (12) and (15) express the exact solution of the linearized problem (6) and (10) for $\sigma^0_{11} = \sigma^0_{22} \neq \sigma^0_{33}$.

The variance tensor components can be calculated by means of the above-mentioned scheme [see (13) and (14)]. Then for $n \neq 0$

TABLE 1

n \ α	0,05	0,1	0,2	0,3	0,4	0,5
0	1,75	3,51	7,02	10,53	14,04	17,55
1	1,13	2,26	4,52	6,78	9,04	11,50
2	0,86	1,72	3,44	5,16	6,88	8,60
3	0,70	1,40	2,80	4,20	5,60	7,00
4	0,59	1,18	2,36	3,54	4,72	5,90
5	0,51	1,02	2,04	3,06	4,08	5,10
6	0,45	0,90	1,80	2,70	3,60	4,50
7	0,40	0,81	1,62	2,43	3,24	4,05
8	0,36	0,73	1,46	2,19	2,92	3,65

$$\begin{aligned}
D_{1111} &= D_{2222} = \frac{\alpha^2 s^{02}}{18n^2}\left[\frac{15B^2-7B+1}{B(4B-1)}+\frac{B(9-30B)}{2(4B-1)}K+\frac{27B-6}{2(4B-1)}L\right],\\
D_{3333} &= \frac{\alpha^2 s^{02}}{9n^2}\left[\frac{10B-1}{2B}-5BK+3{,}5L\right],\\
D_{1122} &= \frac{\alpha^2 s^{02}}{18n^2}\left[\frac{5B^2-5B+1}{B(4B-1)}+\frac{B(3-10B)}{2(4B-1)}K+\frac{25B-6}{2(4B-1)}L\right],\\
D_{1313} &= D_{2323} = \frac{\alpha^2 s^{02}}{18n^2}\left[-5+\frac{10B-1}{2}K-0{,}5L\right],\\
D_{1212} &= \frac{\alpha^2 s^{02}}{18n^2}\left[\frac{5B-1}{4B-1}-\frac{B(10B-3)}{2(4B-1)}K+\frac{B}{2(4B-1)}L\right],\\
D_{1133} &= D_{2233} = \frac{\alpha^2 s^{02}}{9n^2}\left[-2{,}5+2{,}5BK-L\right],
\end{aligned} \tag{16}$$

where

$$B=\frac{1+n}{3n},\quad K=\frac{1}{4\sqrt{B}\sqrt{2\sqrt{B}+1}}\ln\frac{1+\sqrt{2\sqrt{B}+1}+\sqrt{B}}{1-\sqrt{2\sqrt{B}+1}+\sqrt{B}}+\frac{1}{2\sqrt{B}\sqrt{2\sqrt{B}-1}}\left(\operatorname{arctg}\frac{1-\sqrt{B}}{\sqrt{2\sqrt{B}-1}}+\frac{\pi}{2}\right),$$

$$L=-\frac{1}{4\sqrt{2\sqrt{B}+1}}\ln\frac{1+\sqrt{2\sqrt{B}+1}+\sqrt{B}}{1-\sqrt{2\sqrt{B}+1}+\sqrt{B}}+\frac{1}{2\sqrt{2\sqrt{B}-1}}\left(\operatorname{arctg}\frac{1-\sqrt{B}}{\sqrt{2\sqrt{B}-1}}+\frac{\pi}{2}\right).$$

For n = 0 the variance tensor components are expressed by the formulas

$$\begin{aligned}
&D_{1111}=D_{2222}=0.1234\alpha^2 s^{02},\ D_{3333}=0.0730\alpha^2 s^{02},\\
&D_{1122}=0.0984\alpha^2 s^{02},\ D_{1313}=D_{2323}=0.0159\alpha^2 s^{02},\\
&D_{1212}=0.0127\alpha^2 s^{02},\ D_{1133}=D_{2233}=0{,}0063\alpha^2 s^{02}.
\end{aligned} \tag{17}$$

The remaining variance tensor components equal zero.

Values of the quantity $\sqrt{D_{1111}}/s^{\circ}$ (in percents) that characterizes the spread in the stress as a function of the variables α and n are presented in Table 1.

As is known, the power law describes creep well only on a small section of the stress variation. The hyperbolic sine law yields better results for the description of the creep rate dependence on the stress. If it is linearized, then for small stresses $n \approx 0$, and for large n takes on values of the order of 6–8. Consequently, in the high stress domain, where the exponent n = 6–8 corresponds to the creep power law, the relative magnitude of the spread $\sqrt{D_{1111}}/s^{\circ}$ for real materials is between 0.36 (α = 0.05) and 3.65% (α = 0.5). As the stress decreases the values of the creep law exponent also diminish, hence the quantity $\sqrt{D_{1111}}/s^{\circ}$ increases. In the law stress domain where complete physical linearization of the creep law is possible (n = 0), the spread in the stress takes on the greatest value: here $\sqrt{D_{1111}}/s^{\circ}$ is between 1.75 and 17.55%.

Formulas (16) and (17) permit estimation of the magnitude in the spread of the fluctuations σ^*_{33} when $\sigma^{\circ}_{33} = 0$ in the deterministic problem. In this case the spread σ^*_{33} is characterized by the quantity $\sqrt{D_{3333}}/\sigma^{\circ}_{11}$ (for $\sigma^{\circ}_{11} = \sigma^{\circ}_{22}$, $\sigma^{\circ}_{33} = 0 - s^{\circ} = \sigma^{\circ}_{11}$) whose value (in percent) are presented in Table 2 as a function of α and n. The quantity $\sqrt{D_{3333}}/\sigma^{\circ}_{11}$ is approximately just 1.5 times less than the corresponding values of $\sqrt{D_{1111}}/\sigma^{\circ}_{11}$ and consequently, the fluctuations should not be neglected even for $\sigma^{\circ}_{33} = 0$.

An assumption about the smallness of the fluctuation tensor components σ^*_{ij} was made in solving the creep problem for a stochastically inhomogeneous medium. This circumstance

TABLE 2

n \ α	0,05	0,1	0,2	0,3	0,4	0,5
0	1,85	2,70	5,40	8,10	10,80	13,50
1	0,80	1,61	3,22	4,83	6,44	8,05
2	0,58	1,16	2,32	3,48	4,64	5,80
3	0,45	0,91	1,82	2,73	3,64	4,55
4	0,37	0,75	1,50	2,25	3,00	3,75
5	0,32	0,64	1,28	1,92	2,56	3,20
6	0,28	0,56	1,18	1,68	2,24	2,80
7	0,24	0,49	0,98	1,47	1,96	2,45
8	0,22	0,45	0,90	1,25	1,80	2,25

TABLE 3

n \ α	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
1	0,57	1,34	2,28	3,39	4,75	6,38	8,33	10,68	13,78
2	0,79	1,77	2,83	4,07	5,47	7,06	8,86	10,90	13,28
3	0,99	2,29	3,81	5,63	7,74	10,15	12,90	16,05	19,65
4	1,18	2,72	4,61	6,91	9,60	12,74	16,32	20,42	25,10
5	1,28	3,10	5,25	7,95	11,09	14,84	19,14	24,02	29,58
6	1,39	3,30	5,77	8,79	12,40	16,60	21,45	26,97	33,29
7	1,46	3,62	6,14	9,52	13,46	18,06	23,38	29,45	36,36
8	1,52	3,72	6,60	10,31	14,35	19,30	25,01	31,55	38,96

permitted, firstly, neglecting products of the components of the this tensor, and secondly, linearization of the function s^n. Consequently, a statistically and physically linear problem was obtained as a result of these operations.

The errors occurring in the statistical and physical linearizations are interrelated, and consequently obtaining an accurate estimate of the error is not possible as a set. In this connection, approximate error estimates were considered separately for each of the two mentioned types for the case $\overset{\circ}{\sigma}_{11} = \overset{\circ}{\sigma}_{22} \neq \overset{\circ}{\sigma}_{33}$. It is here assumed that the random field $U(x_1, x_2, x_3)$ is normal.

The errors in physical linearization are the result of replacing the nonlinear function s^n by a linear function by expanding it in a power series with subsequent retention of just the linear part of the series [see (7)]. It is here assumed that the principal part of the linearization error is the first term of the discarded series. A relative physical linearization error, f, a random function for which the mathematical expectation and variance were found approximately, was considered.

Errors occur in the statistical linearization because of neglecting products of the form $\alpha U\sigma^*_{ij}$, $\sigma^*_{ij}\sigma^*_{kl}$. These errors were estimated for each component of the creep rate tensor $\dot{\varepsilon}_{ij}$. The mathematical expectation and variance of the relative linearization errors φ_{ij} were calculated. Because of the awkardness the corresponding computations are not presented here.

As an estimate of the relative error in calculating the components of $\dot{\varepsilon}_{ij}$ as a whole, the upper bound of the confidence interval was taken for the quantity $f + \varphi_{ij}$. The fiduciary probability was selected equal to 0.95, while the correlation between errors of two kinds was not taken into account. Values of the estimates for the relative error in calculating the component $\dot{\varepsilon}_{11}$ as a function of α and n (in percents) are presented in Table 3. The error estimates for the other components of $\dot{\varepsilon}_{ij}$ do not exceed the estimates for $\dot{\varepsilon}_{11}$.

From Table 3 it is easy to see that there is a sufficiently broad range of parameters α and n in which the error is completely acceptable for the solution of practical problems. The domain where the error does not exceed 10% is extracted in Table 3 as an example.

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